ATKINSON SCIENCE

THEORY GUIDE

Equations of Fluid Flow

Enthalpy Equation in Cylindrical Coordinates

10 October 2020

Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

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1 Control volume analysis

The energy of a mass of fluid consists of the sum of its internal energy, kinetic energy and potential energy. To derive the energy equation, we begin with the first law of thermodynamics. When applied to the control volume in Figure 1, the first law can be written:

$$\boxed{\begin{array}{l}\text{Rate of increase}\\\text{of energy in CV}\end{array}} = \boxed{\begin{array}{l}\text{Rate of flow of}\\\text{energy into CV}\end{array}} - \boxed{\begin{array}{l}\text{Rate of flow of}\\\text{energy out of CV}\end{array}}$$

$$+ \boxed{\begin{array}{l}\text{Rate of heat}\\\text{transfer into CV}\\\text{by conduction}\end{array}} + \boxed{\begin{array}{l}\text{Rate at which}\\\text{surface and body}\\\text{forces do work}\\\text{on CV}\end{array}} \qquad (1)$$

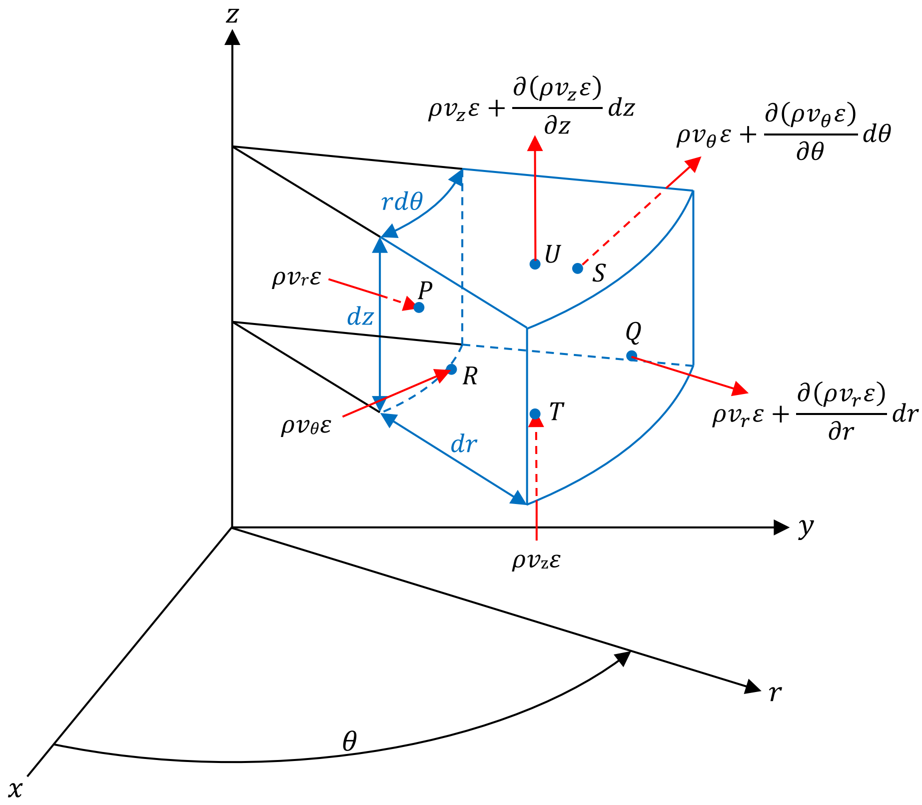
We shall denote the internal energy per unit mass of fluid by e [J kg^{-1}]. The kinetic energy per unit mass K is

$$K = \frac{v_r^2 + v_\theta^2 + v_z^2}{2} \qquad [\mathrm{J\ kg^{-1}}]$$

We shall denote the sum of these components by ε [J kg^{-1}], i.e.

$$\varepsilon = e + \frac{v_r^2 + v_\theta^2 + v_z^2}{2} \qquad [\mathrm{J\ kg^{-1}}]$$

Figure 1 Infinitesimal control volume for cylindrical coordinates



2 Transient and convection terms

The amount of energy in the CV is equal to the energy per unit mass ε [J kg^{-1}] times the mass of fluid in the CV, $\rho\ dr\ rd\theta\ dz$; that is, $\rho\ \varepsilon\ dr\ rd\theta\ dz$ [J]. The rate of increase of energy with time, the left-hand term in Eq. (1), is therefore

$$\frac{\partial(\rho\varepsilon)}{\partial t} dr\ rd\theta\ dz \qquad (2)$$

Energy may enter or leave through any of the faces P to U in Figure 1, transported by the mass flow through the faces.

The rate of flow of energy through the face perpendicular to the r direction whose centre is P is ε [J kg^{-1}] times the mass flow through the face, $\rho\ v_r\ rd\theta\ dz$; that is,

$$\rho v_r \varepsilon\ rd\theta\ dz \qquad [\mathrm{J\ s^{-1}}]$$

The rate of flow of energy through the opposite face whose centre is Q is

$$\left(\rho v_r \varepsilon + \frac{\partial(\rho v_r \varepsilon\ )}{\partial r} dr\right)(r + dr) d\theta\ dz \qquad [\mathrm{J\ s^{-1}}]$$

and so the net rate of flow of energy *out* of the CV through the faces with centres P and Q is

$$\left(\rho v_r \varepsilon + \frac{\partial(\rho v_r \varepsilon)}{\partial r} dr\right)(r + dr) d\theta\ dz - (\rho v_r \varepsilon)\ rd\theta\ dz =$$

$$\frac{\partial(\rho v_r \varepsilon)}{\partial r} dr\ rd\theta\ dz + \rho v_r \varepsilon\ dr\ d\theta\ dz + \frac{\partial(\rho v_r \varepsilon)}{\partial r} dr^2\ d\theta\ dz$$

We can neglect the term in dr^2, so the net rate of flow of energy out of the CV through the faces with centres P and Q is

$$\frac{\partial(\rho v_r \varepsilon)}{\partial r} dr\ rd\theta\ dz + \rho v_r \varepsilon\ dr\ d\theta\ dz$$

The rate of flow of energy through the face perpendicular to the θ direction whose centre is R is ε [J kg^{-1}] times the mass flow through the face, $\rho\ v_\theta\ dr\ dz$; that is,

$$(\rho v_\theta \varepsilon)\ dr\ dz$$

The corresponding rate of flow of energy out of the face with centre S is

$$\left(\rho v_\theta \varepsilon + \frac{\partial(\rho v_\theta \varepsilon)}{\partial \theta} d\theta\right) dr\ dz$$

so the net rate of flow of energy *out* of the CV through the faces with centres R and S is

$$\frac{\partial(\rho v_\theta \varepsilon)}{\partial \theta} dr\ d\theta\ dz$$

The rate of flow of energy through the face perpendicular to the z direction with centre T is ε [J kg^{-1}] times the mass flow through the face. The area of the face is

$$dr\ (r + ½dr)d\theta$$

so the mass flow through the face is

$$(\rho v_z)\ dr\ (r + ½dr)d\theta$$

and the rate of flow of energy through the face is

$$(\rho v_z \varepsilon)\ dr\ (r + ½dr)d\theta$$

The corresponding rate of flow of energy out of the face with centre U is

$$\left(\rho v_z \varepsilon + \frac{\partial(\rho v_z \varepsilon)}{\partial z} dz\right) dr\ (r + ½dr)d\theta$$

so the net rate of flow of energy out of the CV through the faces with centres T and U is

$$\frac{\partial(\rho v_z \varepsilon)}{\partial z} dr\ (r + ½dr)d\theta\ dz = \frac{\partial(\rho v_z \varepsilon)}{\partial z} dr\ rd\theta\ dz + \frac{\partial(\rho v_z \varepsilon)}{\partial z} ½dr^2\ d\theta\ dz$$

We can neglect the term in dr^2, so the net rate of flow of energy out of the CV is

$$\frac{\partial(\rho v_z \varepsilon)}{\partial z} dr\ rd\theta\ dz$$

The sum of the net rates of outflow of energy is

$$\left[\frac{\partial(\rho v_r \varepsilon)}{\partial r} + \frac{\rho v_r \varepsilon}{r} + \frac{1}{r}\frac{\partial(\rho v_\theta \varepsilon)}{\partial \theta} + \frac{\partial(\rho v_z \varepsilon)}{\partial z}\right] dr\ rd\theta\ dz$$

Finally, we can combine the first and second terms in the brackets into one:

$$\left[\frac{1}{r}\frac{\partial(r\rho v_r \varepsilon)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta \varepsilon)}{\partial \theta} + \frac{\partial(\rho v_z \varepsilon)}{\partial z}\right] dr\ rd\theta\ dz \qquad (3)$$

3 Heat transfer term

The third term on the right of (1) represents the heat transfer into the CV by conduction. We shall denote the heat flux per unit area by $\boldsymbol{q}$ [J m^{-2} s^{-1}]. $\boldsymbol{q}$ has components q_r, q_θ, and q_z [J m^{-2} s^{-1}] in the r, θ and z directions, respectively. Heat transfer is considered positive if it is in the positive coordinate direction.

Referring to Figure 2, the rate of heat flow through the face perpendicular to the r direction whose centre is P is q_r [J m^{-2} s^{-1}] times the area of the face, $rd\theta\ dz$ [m^2]; that is,

$$q_r\ rd\theta\ dz$$

The rate of heat flow through the opposite face whose centre is Q is

$$\left(q_r + \frac{\partial q_r}{\partial r} dr\right)(r + dr)d\theta\ dz$$

and so the net rate of heat flow *out* of the CV through the faces with centres P and Q is

$$\left(q_r + \frac{\partial q_r}{\partial r} dr\right)(r + dr)d\theta\ dz - q_r\ rd\theta\ dz$$

$$\left(q_r + \frac{\partial q_r}{\partial r} dr\right) rd\theta\ dz + \left(q_r + \frac{\partial q_r}{\partial r} dr\right) dr\ d\theta\ dz - q_r\ rd\theta\ dz$$

$$= \frac{\partial q_r}{\partial r} dr\ rd\theta\ dz + \ q_r\ dr\ d\theta\ dz + \frac{\partial q_r}{\partial r} dr^2\ d\theta\ dz$$

We can neglect the term in dr^2, so the net rate of heat flow out of the CV through the faces with centres P and Q is

$$\frac{\partial q_r}{\partial r} dr\ rd\theta\ dz + q_r\ dr\ d\theta\ dz \qquad [\mathrm{J\ s^{-1}}]$$

The rate of heat flow through the face normal to the θ direction with centre R is q_θ [J m^{-2} s^{-1}] times the area of the face, $dr\ dz$ [m^2]; that is,

$$q_\theta\ dr\ dz$$

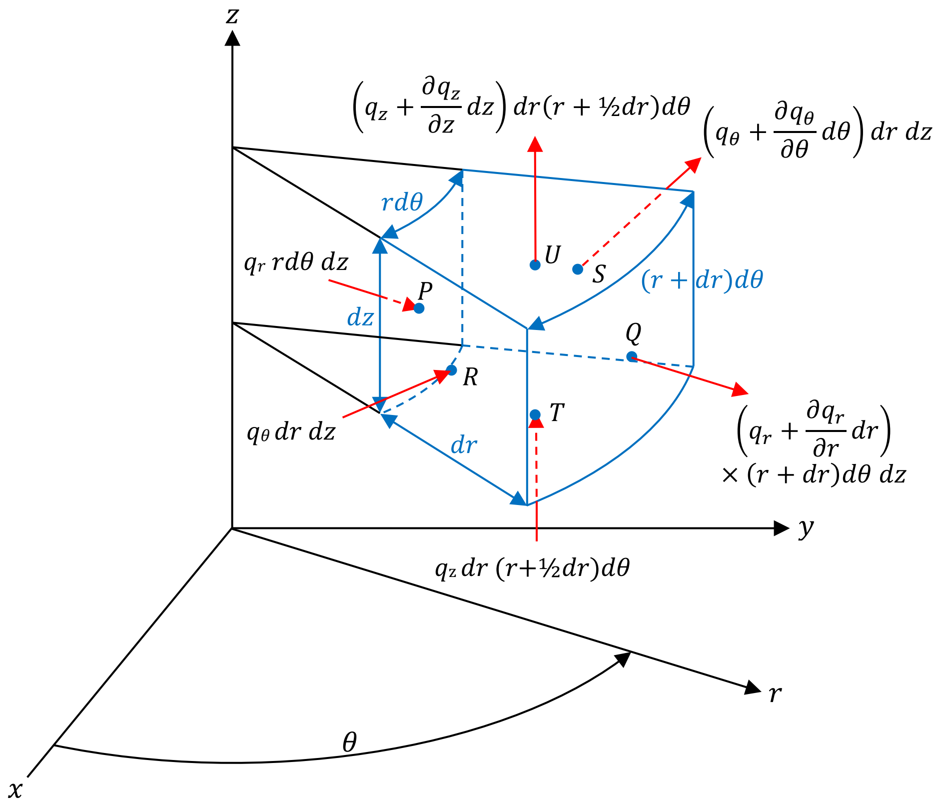
The corresponding rate of heat flow out of the face with centre S is

$$\left(q_\theta + \frac{\partial q_\theta}{\partial \theta} d\theta\right) dr\ dz$$

so the net rate of heat flow out of the CV through the faces with centres R and S is

$$\frac{\partial q_\theta}{\partial \theta} dr\ d\theta\ dz \qquad [\mathrm{J\ s^{-1}}]$$

Figure 2 Heat flows in the r, θ and z directions



The rate of heat flow through the face normal to the z direction with centre T is q_z [J m^{-2} s^{-1}] times the area of the face. The area of the face is

$$dr\,(r + ½dr)d\theta$$

so the rate of heat flow through the face is

$$q_z\, dr\,(r + ½dr)d\theta$$

The corresponding rate of flow of mass out of the face with centre U is

$$\left(q_z + \frac{\partial q_z}{\partial z} dz\right) dr\,(r + ½dr)d\theta$$

so the net rate of heat flow out of the CV through the faces with centres T and U is

$$\frac{\partial q_z}{\partial z} dr\,(r + ½dr)d\theta\, dz = \frac{\partial q_z}{\partial z} dr\, rd\theta\, dz + \frac{\partial q_z}{\partial z} ½dr^2\, d\theta\, dz$$

We can neglect the term in dr^2, so the net rate of flow out of the CV is

$$\frac{\partial q_z}{\partial z} dr\, rd\theta\, dz \quad [\text{kg s}^{-1}]$$

The rate of heat flow *into* the CV, the third term on the right of (1), is therefore

$$-\left(\frac{\partial q_r}{\partial r}+\frac{q_r}{r}+\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{\partial q_z}{\partial z}\right) dr\, rd\theta\, dz \qquad [\mathrm{J\ s^{-1}}]$$

We can combine the first two terms in the brackets into one, so the rate of heat flow into the CV is

$$-\left(\frac{1}{r}\frac{\partial r q_r}{\partial r}+\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{\partial q_z}{\partial z}\right) dr\, rd\theta\, dz \qquad [\mathrm{J\ s^{-1}}]$$

The heat flux components q_r, q_θ, q_z are

$$q_r=-k\frac{\partial T}{\partial r}, \quad q_\theta=-\frac{k}{r}\frac{\partial T}{\partial \theta}, \quad q_z=-k\frac{\partial T}{\partial z}$$

where k [W m^{-1} K^{-1}] is the thermal conductivity of the fluid. We can now write the heat flow into the CV, the third term on the right of (1), in terms of temperature:

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] dr\, rd\theta\, dz \qquad [\mathrm{J\ s^{-1}}] \qquad (4)$$

4 Pressure work term

The rate at which pressure does work on one side of a flat moving fluid surface is the product of the pressure, the area of the surface, and the component of velocity normal to the surface. By definition, a positive pressure acts inward. Referring to Figure 3, the rate at which work is done on the fluid that enters the CV through the face perpendicular to the r direction whose centre is P is

$$pv_r \, rd\theta \, dz$$

The rate at which work is done on the fluid that leaves the CV through the face perpendicular to the r direction whose centre is Q is

$$-\left[pv_r + \frac{\partial(pv_r)}{\partial r}dr\right](r + dr)d\theta \, dz$$

$$= -\left[pv_r + \frac{\partial(pv_r)}{\partial r}dr\right]rd\theta \, dz - \left[pv_r + \frac{\partial(pv_r)}{\partial r}dr\right]dr \, d\theta \, dz$$

$$= -pv_r \, rd\theta \, dz - \frac{\partial(pv_r)}{\partial r}dr \, rd\theta \, dz - pv_r \, dr \, d\theta \, dz$$

after neglecting the term in $(dr)^2$. The net pressure work associated with the two faces normal to the r direction is

$$-\frac{\partial(pv_r)}{\partial r}dr \, rd\theta \, dz - pv_r \, dr \, d\theta \, dz \quad [\mathrm{J\ s^{-1}}]$$

The rate at which work is done on the fluid that enters the CV through the face perpendicular to the θ direction whose centre is R is

$$pv_\theta \, dr \, dz$$

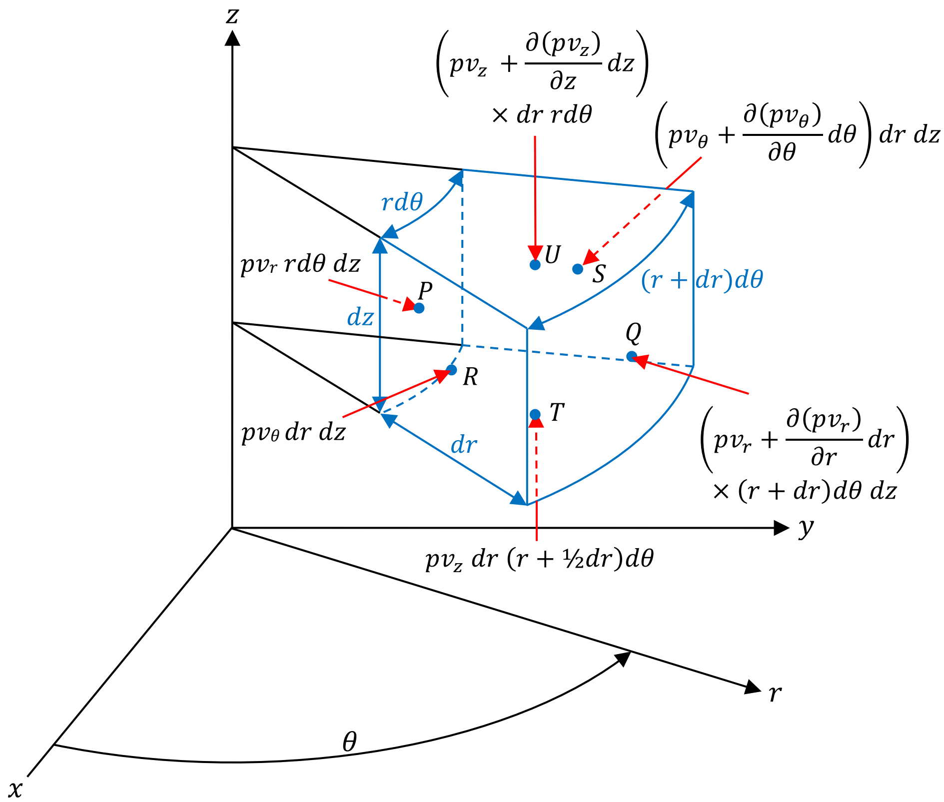
The rate at which work is done on the fluid that leaves the CV through the face perpendicular to the θ direction whose centre is T is

$$-\left[pv_\theta + \frac{\partial(pv_\theta)}{\partial\theta}d\theta\right]dr \, dz$$

The net pressure work associated with the two faces normal to the θ direction is

$$-\frac{\partial(pv_\theta)}{\partial\theta}dr \, d\theta \, dz \quad [\mathrm{J\ s^{-1}}]$$

Figure 3 Pressure work done in the r, θ and z directions



The rate at which work is done on the fluid that enters the CV through the face perpendicular to the z direction with centre at T is ρv_z times the area of the face. The area of the face is

$$dr\,(r + ½dr)d\theta$$

so the rate of work done is

$$pv_z\, dr\,(r + ½dr)d\theta$$

The rate at which work is done on the fluid that leaves the CV through the face perpendicular to the z direction whose centre is U is

$$-\left[pv_z + \frac{\partial(pv_z)}{\partial z}dz\right] dr\,(r + ½dr)d\theta$$

$$= -\left[pv_z + \frac{\partial(pv_z)}{\partial z}dz\right] dr\, rd\theta - \left[pv_z + \frac{\partial(pv_z)}{\partial z}dz\right] ½(dr)^2 d\theta$$

We can neglect the term in $(dr)^2$, so the work done on the fluid that leaves the CV is

$$-\left[pv_z + \frac{\partial(pv_z)}{\partial z}dz\right] dr\, rd\theta$$

The net pressure work associated with the two faces normal to the z direction is

$$-\left[pv_z + \frac{\partial(pv_z)}{\partial z}dz\right]dr\,rd\theta + pv_z\,dr\,(r + ½dr)d\theta$$

$$= -pv_z\,dr\,rd\theta\,dz - \frac{\partial(pv_z)}{\partial z}dr\,rd\theta\,dz + pv_z\,dr\,rd\theta + pv_z½(dr)^2 d\theta$$

$$-\frac{\partial(pv_z)}{\partial z}dr\,rd\theta\,dz \qquad [\mathrm{J\,s^{-1}}]$$

after neglecting the term in $(dr)^2$.

After adding together the terms for the three pairs of faces, the net pressure work done on the fluid in the CV, the fourth term on the right of (1), is

$$-\left[\frac{\partial(pv_r)}{\partial r} + \frac{pv_r}{r} + \frac{1}{r}\frac{\partial(pv_\theta)}{\partial\theta} + \frac{\partial(pv_z)}{\partial z}\right]dr\,rd\theta\,dz$$

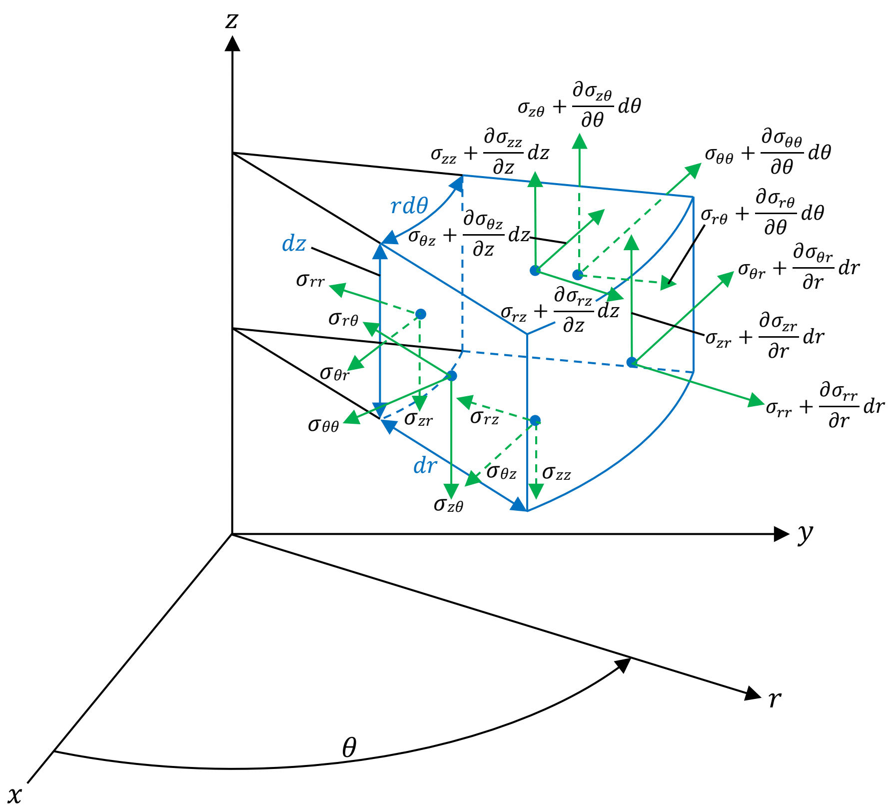
We can combine the first two terms in the brackets, so the net pressure work done on the fluid in the CV is

$$-\left[\frac{1}{r}\frac{\partial(rpv_r)}{\partial r} + \frac{1}{r}\frac{\partial(pv_\theta)}{\partial\theta} + \frac{\partial(pv_z)}{\partial z}\right]dr\,rd\theta\,dz \qquad [\mathrm{J\,s^{-1}}] \qquad (5)$$

5 Work done by viscous stresses

If a fluid element changes size or shape with time, viscosity creates further stresses that may act normal to a surface (a viscous normal stress) or tangentially (a viscous shear stress). We define the different components of viscous normal stress and viscous shear stress as shown in Figure 4. The first subscript of the symbol σ represents the direction of the stress and the second subscript represents the direction of the surface normal.

Figure 4 Viscous stresses on the control volume



5.1 Work done by normal stresses

By convention, a normal stress is positive if it acts *outwards* from the CV (in contrast with p, which is positive inwards). The rate of work done by the normal stresses σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} on the fluid in the CV can be found in the same way as for p, remembering the change of sign. Thus the rate of work done by the normal stresses is

$$\left[\frac{1}{r}\frac{\partial(r\sigma_{rr}v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\sigma_{\theta\theta}v_\theta)}{\partial \theta} + \frac{\partial(\sigma_{zz}v_z)}{\partial z}\right] dr\, rd\theta\, dz \quad [\mathrm{J\,s^{-1}}] \quad (6)$$

5.2 Work done by shear stresses

By convention, the shear stresses are taken as positive on the faces farthest from the origin. Thus a shear stress $\sigma_{\theta r}$ acts in the positive θ direction on the visible (upper) face perpendicular to the r axis and a corresponding shear stress acts in the negative θ direction on the invisible (lower) face perpendicular to the r axis.

Referring to Figure 4, the shear stress acting in the θ direction on the *lower* face normal to the r direction is $-\sigma_{\theta r}$ and the rate at which work is done by this stress is

$$-\sigma_{\theta r} v_\theta \, r d\theta \, dz \qquad [\mathrm{J\ s^{-1}}]$$

The shear force acting in the θ direction on the *upper* face normal to the r direction is

$$\left(\sigma_{\theta r} + \frac{\partial \sigma_{\theta r}}{\partial r} dr\right)(r + dr) d\theta \, dz \qquad [\mathrm{N}]$$

and the rate at which work is done by this stress is

$$\left(\sigma_{\theta r} + \frac{\partial \sigma_{\theta r}}{\partial r} dr\right)\left(v_\theta + \frac{\partial v_\theta}{\partial r} dr\right)(r + dr) d\theta \, dz$$

$$= \left[\sigma_{\theta r} v_\theta + \sigma_{\theta r} \frac{\partial v_\theta}{\partial r} dr + v_\theta \frac{\partial \sigma_{\theta r}}{\partial r} dr + \frac{\partial \sigma_{\theta r}}{\partial r} \frac{\partial v_\theta}{\partial r} (dr)^2\right](r + dr) d\theta \, dz$$

$$= \left(\sigma_{\theta r} v_\theta r + \sigma_{\theta r} v_\theta dr + \sigma_{\theta r} \frac{\partial v_\theta}{\partial r} r dr + v_\theta \frac{\partial \sigma_{\theta r}}{\partial r} r dr\right) d\theta \, dz \qquad [\mathrm{J\ s^{-1}}]$$

after neglecting the term in $(dr)^2$. The net work done by the shear stress $\sigma_{\theta r}$ on the two faces normal to the r direction is

$$\left(\sigma_{\theta r} v_\theta r + \sigma_{\theta r} v_\theta dr + \sigma_{\theta r} \frac{\partial v_\theta}{\partial r} r dr + v_\theta \frac{\partial \sigma_{\theta r}}{\partial r} r dr\right) d\theta \, dz - \sigma_{\theta r} v_\theta \, r d\theta \, dz$$

$$= \left(\frac{\sigma_{\theta r} v_\theta}{r} + \frac{\partial (\sigma_{\theta r} v_\theta)}{\partial r}\right) dr \, r d\theta \, dz$$

$$= \frac{1}{r} \frac{\partial (r \sigma_{\theta r} v_\theta)}{\partial r} \, dr \, r d\theta \, dz \qquad [\mathrm{J\ s^{-1}}]$$

There is also a shear stress σ_{zr} on these two faces, and the net work done by this shear stress is

$$= \frac{1}{r} \frac{\partial (r \sigma_{zr} v_z)}{\partial r} \, dr \, r d\theta \, dz \qquad [\mathrm{J\ s^{-1}}]$$

The net work done by shear stresses on the two faces normal to the r direction is therefore

$$\left[\frac{1}{r} \frac{\partial (r \sigma_{\theta r} v_\theta)}{\partial r} + \frac{1}{r} \frac{\partial (r \sigma_{zr} v_z)}{\partial r}\right] dr \, r d\theta \, dz \qquad [\mathrm{J\ s^{-1}}]$$

Referring to Figure 4, the shear stress acting in the r direction on the *lower* face normal to the θ direction is $-\sigma_{r\theta}$ and the rate at which work is done by this stress is

$$-\sigma_{r\theta} v_r \, dr \, dz \qquad [\mathrm{J\ s^{-1}}]$$

The shear force acting in the r direction on the *upper* face normal to the θ direction is

$$\left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta} d\theta\right) dr \, dz \qquad [\mathrm{N}]$$

and the rate at which work is done by this stress is

$$\left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta} d\theta\right)\left(v_r + \frac{\partial v_r}{\partial \theta} d\theta\right) dr \, dz$$

$$= \left(\sigma_{r\theta} v_r + \sigma_{r\theta} \frac{\partial v_r}{\partial \theta} d\theta + v_r \frac{\partial \sigma_{r\theta}}{\partial \theta} d\theta + \frac{\partial \sigma_{r\theta}}{\partial \theta}\frac{\partial v_r}{\partial \theta} (d\theta)^2\right) dr \, dz$$

$$= \left(\sigma_{r\theta} v_r + \frac{\partial(\sigma_{r\theta} v_r)}{\partial \theta} d\theta\right) dr \, dz$$

after neglecting the term in $(d\theta)^2$. The net work done by the shear stress $\sigma_{r\theta}$ on the two faces normal to the θ direction is

$$\left(\sigma_{r\theta} v_r + \frac{\partial(\sigma_{r\theta} v_r)}{\partial \theta} d\theta\right) dr \, dz - \sigma_{r\theta} v_r \, dr \, dz$$

$$\frac{1}{r}\frac{\partial(\sigma_{r\theta} v_r)}{\partial \theta} dr \, r d\theta \, dz \qquad [\mathrm{J\ s^{-1}}]$$

There is also a shear stress $\sigma_{z\theta}$ on these two faces, and the net work done by this shear stress is

$$= \frac{1}{r}\frac{\partial(\sigma_{z\theta} v_z)}{\partial \theta} \, dr \, r d\theta \, dz \qquad [\mathrm{J\ s^{-1}}]$$

The net work done by shear stresses on the two faces normal to the θ direction is therefore

$$\left[\frac{1}{r}\frac{\partial(\sigma_{r\theta} v_r)}{\partial \theta} + \frac{1}{r}\frac{\partial(\sigma_{z\theta} v_z)}{\partial \theta}\right] dr \, r d\theta \, dz \qquad [\mathrm{J\ s^{-1}}]$$

Referring to Figure 4, the shear stress acting in the r direction on the *lower* face normal to the z direction is $-\sigma_{rz}$ and the rate at which work is done by this stress is

$$-\sigma_{rz}v_r\, dr\, (r + ½dr)d\theta$$

$$= -\sigma_{rz}v_r\, dr\, rd\theta - \sigma_{rz}v_r\, ½(dr)^2 d\theta$$

$$= -\sigma_{rz}v_r\, dr\, rd\theta \quad [\mathrm{J\, s^{-1}}]$$

after neglecting the term in $(dr)^2$.

The shear force acting in the r direction on the *upper* face normal to the z direction is

$$\left(\sigma_{rz} + \frac{\partial \sigma_{rz}}{\partial z}dz\right) dr\, (r + ½dr)d\theta \quad [\mathrm{N}]$$

and the rate at which work is done by this stress is

$$\left(\sigma_{rz} + \frac{\partial \sigma_{rz}}{\partial z}dz\right)\left(v_r + \frac{\partial v_r}{\partial z}dz\right) dr\, (r + ½dr)d\theta$$

$$= \left(\sigma_{rz}v_r + \sigma_{rz}\frac{\partial v_r}{\partial z}dz + \frac{\partial \sigma_{rz}}{\partial z}dz + \frac{\partial \sigma_{rz}}{\partial z}\frac{\partial v_r}{\partial z}(dz)^2\right) dr\, (r + ½dr)d\theta$$

$$= \left(\sigma_{rz}v_r + \frac{\partial(\sigma_{rz}v_r)}{\partial z}dz\right) dr\, rd\theta \quad [\mathrm{J\, s^{-1}}]$$

after neglecting terms in $(dz)^2$ and $(dr)^2$. The net work done by the shear stress σ_{rz} on the two faces normal to the z direction is

$$\left(\sigma_{rz}v_r + \frac{\partial(\sigma_{rz}v_r)}{\partial z}dz\right) dr\, rd\theta - \sigma_{rz}v_r\, dr\, rd\theta$$

$$= \frac{\partial(\sigma_{rz}v_r)}{\partial z} dr\, rd\theta\, dz \quad [\mathrm{J\, s^{-1}}]$$

There is also a shear stress $\sigma_{\theta z}$ on these two faces, and the net work done by this shear stress is

$$= \frac{\partial(\sigma_{\theta z}v_\theta)}{\partial z}\, dr\, rd\theta\, dz \quad [\mathrm{J\, s^{-1}}]$$

The net work done by shear stresses on the two faces normal to the z direction is therefore

$$\left[\frac{\partial(\sigma_{rz}v_r)}{\partial z} + \frac{\partial(\sigma_{\theta z}v_\theta)}{\partial z}\right] dr\, rd\theta\, dz \quad [\mathrm{J\, s^{-1}}]$$

The net work done by shear stresses on the fluid in the CV, is therefore

$$\left[\frac{1}{r}\frac{\partial(r\sigma_{\theta r}v_\theta)}{\partial r} + \frac{1}{r}\frac{\partial(r\sigma_{zr}v_z)}{\partial r} + \frac{1}{r}\frac{\partial(\sigma_{r\theta}v_r)}{\partial \theta} + \frac{1}{r}\frac{\partial(\sigma_{z\theta}v_z)}{\partial \theta} + \frac{\partial(\sigma_{rz}v_r)}{\partial z} + \frac{\partial(\sigma_{\theta z}v_\theta)}{\partial z}\right] dr\, rd\theta\, dz \qquad (7)$$

6 Body force terms

The simplest example of a body force is the gravitational force. The fluid in the CV is subject to a gravitational force equal to the mass of the fluid $\rho\ dr\ rd\theta\ dz$ times the acceleration due to gravity g [m s^{-1}]; that is, $\rho\ g\ dr\ rd\theta\ dz$ [kg m s^{-1}]. A body force is a vector, so in general it has three components, f_r, f_θ, f_z per unit mass [m s^{-1}]. The body forces acting in the r, θ and z coordinate directions are, respectively,

$$\rho f_r\, dr\ rd\theta\ dz, \quad \rho f_\theta\, dr\ rd\theta\ dz, \quad \rho f_z\, dr\ rd\theta\ dz$$

The rate of work done by the body forces on the fluid in the CV is simply work = force × velocity; that is,

$$(v_r f_r + v_\theta f_\theta + v_z f_z)\rho\ dr\ rd\theta\ dz \qquad (8)$$

7 Energy equation in terms of stress

Substituting the terms (2), (3), (4), (5), (6), (7) and (8) into Eq. (1) and dividing by $dr\ rd\theta\ dz$ gives the energy equation:

$$\frac{\partial(\rho\varepsilon)}{\partial t} = -\left[\frac{1}{r}\frac{\partial(r\rho v_r\varepsilon)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta\varepsilon)}{\partial\theta} + \frac{\partial(\rho v_z\varepsilon)}{\partial z}\right]$$

$$+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(k\frac{\partial T}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]$$

$$-\left[\frac{1}{r}\frac{\partial(rpv_r)}{\partial r} + \frac{1}{r}\frac{\partial(pv_\theta)}{\partial\theta} + \frac{\partial(pv_z)}{\partial z}\right]$$

$$+\left[\frac{1}{r}\frac{\partial(r\sigma_{rr}v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\sigma_{\theta\theta}v_\theta)}{\partial\theta} + \frac{\partial(\sigma_{zz}v_z)}{\partial z}\right]$$

$$+\left[\frac{1}{r}\frac{\partial(r\sigma_{\theta r}v_\theta)}{\partial r} + \frac{1}{r}\frac{\partial(r\sigma_{zr}v_z)}{\partial r} + \frac{1}{r}\frac{\partial(\sigma_{r\theta}v_r)}{\partial\theta} + \frac{1}{r}\frac{\partial(\sigma_{z\theta}v_z)}{\partial\theta} + \frac{\partial(\sigma_{rz}v_r)}{\partial z} + \frac{\partial(\sigma_{\theta z}v_\theta)}{\partial z}\right]$$

$$\rho[v_r f_r + v_\theta f_\theta + v_z f_z] \quad [\mathrm{J\ s^{-1}}] \quad (9)$$

where the energy per unit mass ε [J kg^{-1}] is

$$\varepsilon = e + \frac{v_r^2 + v_\theta^2 + v_z^2}{2} \qquad [\mathrm{J\ kg^{-1}}]$$

8 Equation for internal energy

We now have an equation for the internal and kinetic energy in a three-dimensional, unsteady, compressible fluid flow. To obtain an equation for the internal energy e alone, we must subtract out the kinetic energy K terms

$$\frac{\partial(\rho K)}{\partial t}+\frac{1}{r}\frac{\partial(r\rho v_r K)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta K)}{\partial \theta}+\frac{\partial(\rho v_z K)}{\partial z}$$

from the energy equation.

In Ref. [1] we derived the conservation equation for K in cylindrical coordinates:

$$\frac{\partial \rho K}{\partial t}+\frac{1}{r}\frac{\partial(r\rho v_r K)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta K)}{\partial \theta}+\rho v_z\frac{\partial(\rho v_z K)}{\partial z}$$

$$=-\left[v_r\frac{\partial p}{\partial r}+\frac{v_\theta}{r}\frac{\partial p}{\partial \theta}+v_z\frac{\partial p}{\partial z}\right]$$

$$+v_r\left[\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_{rr}}{r}-\frac{\sigma_{\theta\theta}}{r}\right]+v_\theta\left[\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\sigma_{\theta r}}{r}+\frac{\partial \sigma_{\theta r}}{\partial r}+\frac{\partial \sigma_{\theta z}}{\partial z}+\frac{\sigma_{r\theta}}{r}\right]$$

$$+v_z\left[\frac{1}{r}\frac{\partial r\sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}\right]$$

$$\rho v_r f_r+\rho v_\theta f_\theta+\rho v_z f_z \quad [\mathrm{J\ kg^{-1}\ s^{-1}}] \qquad (10)$$

Subtracting (10) from (9) gives

$$\frac{\partial(\rho e)}{\partial t}=-\left[\frac{1}{r}\frac{\partial(r\rho v_r e)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta e)}{\partial \theta}+\frac{\partial(\rho v_z e)}{\partial z}\right]$$

$$+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]$$

$$-\left[\frac{1}{r}\frac{\partial(rpv_r)}{\partial r}+\frac{1}{r}\frac{\partial(pv_\theta)}{\partial \theta}+\frac{\partial(pv_z)}{\partial z}\right]+\left[v_r\frac{\partial p}{\partial r}+\frac{v_\theta}{r}\frac{\partial p}{\partial \theta}+v_z\frac{\partial p}{\partial z}\right]+\Theta$$

where Θ is the viscous dissipation term.

This equation simplifies to

$$\frac{\partial(\rho e)}{\partial t}=-\left[\frac{1}{r}\frac{\partial(r\rho v_r e)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta e)}{\partial \theta}+\frac{\partial(\rho v_z e)}{\partial z}\right]$$

$$+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)\right]$$

$$-p\left[\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}\right]+\Theta \qquad (11)$$

9 Viscous dissipation term

The viscous dissipation term in (11) is

$$\Theta = \left[\frac{1}{r}\frac{\partial(r\sigma_{rr}v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\sigma_{\theta\theta}v_\theta)}{\partial\theta} + \frac{\partial(\sigma_{zz}v_z)}{\partial z}\right]$$

$$+\left[\frac{1}{r}\frac{\partial(r\sigma_{\theta r}v_\theta)}{\partial r} + \frac{1}{r}\frac{\partial(r\sigma_{zr}v_z)}{\partial r} + \frac{1}{r}\frac{\partial(\sigma_{r\theta}v_r)}{\partial\theta} + \frac{1}{r}\frac{\partial(\sigma_{z\theta}v_z)}{\partial\theta} + \frac{\partial(\sigma_{rz}v_r)}{\partial z} + \frac{\partial(\sigma_{\theta z}v_\theta)}{\partial z}\right]$$

$$-v_r\left[\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr}}{r} - \frac{\sigma_{\theta\theta}}{r}\right] - v_\theta\left[\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\sigma_{\theta r}}{r} + \frac{\partial\sigma_{\theta r}}{\partial r} + \frac{\partial\sigma_{\theta z}}{\partial z} + \frac{\sigma_{r\theta}}{r}\right]$$

$$-v_z\left[\frac{1}{r}\frac{\partial(r\sigma_{zr})}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z}\right]$$

This equation simplifies to

$$\Theta = \sigma_{rr}\frac{\partial v_r}{\partial r} + \frac{\sigma_{\theta\theta}}{r}\frac{\partial v_\theta}{\partial\theta} + \sigma_{zz}\frac{\partial v_z}{\partial z}$$

$$+\frac{\sigma_{r\theta}}{r}\frac{\partial v_r}{\partial\theta} + \sigma_{rz}\frac{\partial v_r}{\partial z} + \sigma_{\theta r}\frac{\partial v_\theta}{\partial r} - \frac{\sigma_{\theta r}v_\theta}{r} + \sigma_{\theta z}\frac{\partial v_\theta}{\partial z} + \sigma_{zr}\frac{\partial v_z}{\partial r} + \frac{\sigma_{z\theta}}{r}\frac{\partial v_z}{\partial\theta} + \frac{\sigma_{\theta\theta}v_r}{r} \qquad (12)$$

In cylindrical coordinates the stress terms are

$$\sigma_{rr} = 2\mu\frac{\partial v_r}{\partial r} + \lambda\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}\right)$$

$$\sigma_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right) + \lambda\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}\right)$$

$$\sigma_{zz} = 2\mu\frac{\partial v_z}{\partial z} + \lambda\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}\right)$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

$$\sigma_{rz} = \sigma_{zr} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

$$\sigma_{\theta z} = \sigma_{z\theta} = \mu\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} + \frac{\partial v_\theta}{\partial z}\right)$$

Substituting the stress equations into (12) gives

$$\Theta = 2\mu\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + 2\mu\left(\frac{\partial v_z}{\partial z}\right)^2$$

$$+\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2 + \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2 + \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2$$

$$+\lambda\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right)^2 \qquad (13)$$

Substituting (13) into (11) gives the equation for internal energy e alone in cylindrical coordinates in terms of temperature and velocity:

$$\frac{\partial(\rho e)}{\partial t} = -\left[\frac{1}{r}\frac{\partial(r\rho v_r e)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta e)}{\partial \theta} + \frac{\partial(\rho v_z e)}{\partial z}\right]$$

$$+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]$$

$$-p\left[\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right]$$

$$+ 2\mu\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + 2\mu\left(\frac{\partial v_z}{\partial z}\right)^2$$

$$+\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2 + \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2 + \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2$$

$$+\lambda\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right)^2 \qquad (14)$$

10 Enthalpy equation

The enthalpy per unit mass h is defined by

$$h = e + \frac{p}{\rho}$$

To obtain the conservation equation for h, we need to add

$$\frac{\partial\left(\rho\frac{p}{\rho}\right)}{\partial t} + \frac{1}{r}\frac{\partial\left(r\rho v_r\frac{p}{\rho}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\rho v_\theta\frac{p}{\rho}\right)}{\partial \theta} + \frac{\partial\left(\rho v_z\frac{p}{\rho}\right)}{\partial z} = \frac{\partial p}{\partial t} + \frac{1}{r}\frac{\partial(r v_r p)}{\partial r} + \frac{1}{r}\frac{\partial(v_\theta p)}{\partial \theta} + \frac{\partial(v_z p)}{\partial z}$$

to both sides of (14). This gives

$$\frac{\partial(\rho h)}{\partial t} = -\left[\frac{1}{r}\frac{\partial(r\rho v_r h)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta h)}{\partial \theta} + \frac{\partial(\rho v_z h)}{\partial z}\right]$$

$$+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]$$

$$+\frac{\partial p}{\partial t} + v_r\frac{\partial p}{\partial r} + \frac{v_\theta}{r}\frac{\partial p}{\partial \theta} + v_z\frac{\partial p}{\partial z}$$

$$+ 2\mu\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + 2\mu\left(\frac{\partial v_z}{\partial z}\right)^2$$

$$+\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2 + \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2 + \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2$$

$$+\lambda\left(\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right)^2 \qquad (15)$$

which is the conservation equation for the enthalpy h in terms of temperature and velocity for a three-dimensional, unsteady, compressible fluid flow.

11 References

1. K. N. Atkinson, *Equations of Fluid Flow, Kinetic Energy Equation in Cartesian and Cylindrical Coordinates, Theory Guide*, Atkinson Science Limited, 2020.